

Integration Measure and Spectral Flow in the Critical $N=2$ String

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I present the moduli space of the $(2+2)$ -dimensional critical closed fermionic string with two world-sheet supersymmetries. The integration of fermionic and Maxwell moduli in the presence of punctures yields the string measure for n -point amplitudes at arbitrary genus and instanton number. Generalized picture-changing and spectral-flow operators emerge, connecting different instanton sectors. Tree and loop amplitudes are computed.

1. $N=2$ Strings

I am reporting on new results obtained in collaboration with Jan Bischoff, a PhD student, and Sergei Ketov, postdoc at Hannover. Since I have to be brief today, let me refer to our preprints for more details [1–5]. When I talk about $N=2$ strings, I mean fermionic strings with two world-sheet supersymmetries. Those strings are known to be critical in flat complex spacetime $\mathbf{C}^{1,1}$. In this talk I shall consider only untwisted, oriented, closed strings in flat space and will outline their covariant path-integral quantization à la BRST in the (old-fashioned) NSR description. Most concepts and techniques should appear familiar to string folks from ten years ago.

Strings with extended supersymmetry were invented in 1976 [6,7]. It took a while to realize that their critical flat spacetime is *complex* two-dimensional [8,9]. The spectral flow automorphism of the $N=2$ superconformal algebra was discovered in 1987 [10]. $N=2$ strings have only a *finite* number of massless physical excitations, but satisfy duality nevertheless by killing all amplitudes with more than three legs. This curious fact was more or less established by Ooguri and Vafa [11]. A first partial BRST analysis of the spectrum was performed in 1992 [12]. Subsequently, Siegel made some interesting conjectures about connections to the $N=4$ string and the possibility of spacetime supersymmetry in these theories [13]. Lately, there has been an increase of activity in the field (see, e.g., [14]), triggered by the

embedding of bosonic and $N=1$ strings into $N=2$ strings [15–18] and by a new formulation using a topologically twisted $N=4$ string [19,20]. For older reviews on the subject, consult refs. [21,22].

Let me begin by listing the world-sheet degrees of freedom and their symmetries. The starting point is the $N=2$ world-sheet supergravity action [23]. The extended supergravity multiplet involves the real metric $h_{\alpha\beta}$, a *complex* gravitino χ_α and a real Maxwell field (graviphoton) A_α , with $\alpha = 0, 1$. The conformal matter consists of two complex string coordinates X^μ and two complex spin $1/2$ (NSR) fermions ψ^μ , with $\mu = 0, 1$.

The supergravity action is invariant under $N=2$ super-diffeomorphisms and $N=2$ super-Weyl transformations, but shows also a global target space symmetry generated by rigid translations and a complex ‘Lorentz group’ of $U(1,1) \times \mathbf{Z}_2$ [11]. Superconformal gauge-fixing leads to the appropriate set of chiral ghosts, namely a real pair of anticommuting reparametrization ghosts (b, c) , a complex pair of commuting supersymmetry ghosts (β, γ) , and a real pair of anticommuting Maxwell ghosts (\tilde{b}, \tilde{c}) , plus their antichiral partners. The residual $N=2$ superconformal symmetry is generated by the currents (T, G^\pm, J) . As usual, world-sheet fermions may be periodic or antiperiodic around the cylinder. Due to their complex nature, however, the spin fields mapping Neveu-Schwarz to Ramond states carry *double* spinor indices, thus qualifying as spacetime bosons. Hence, there is no obvious room for spacetime supersymmetry.

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2. $N=2$ Moduli

$N=2$ super Riemann surfaces are topologically classified by two invariants, namely the Euler number, $\chi = 2-2g = \frac{1}{2\pi} \int R$ where $g \in \mathbf{Z}_+$ is the genus and R denotes the curvature two-form, as well as the instanton number or monopole charge, $c = \frac{1}{2\pi} \int F \in \mathbf{Z}$, with F being the field-strength two-form. Since the total conformal and $U(1)$ anomalies disappear for $d_C=2$, the local symmetries may be used to eliminate all but finitely many $N=2$ supergravity degrees of freedom for any given pair (g, c) . What remains is the integration over $N=2$ supermoduli space, parametrized $m = (m_h, m_\chi, m_A)$ by metric, fermionic, and Maxwell moduli.

Any metric may be decomposed into a background part carrying the Euler number, two parts given by a diffeomorphism and a Weyl transformation, plus a moduli component, $h(m_h)$. In complex count, the metric moduli space is spanned by $3g-3$ holomorphic quadratic differentials h_ℓ , which are dual to the Beltrami differentials. The latter may be generated by quasi-conformal vector fields having jump discontinuities along (a maximal set of) closed, non-intersecting, minimal geodesics Γ_ℓ .

The fermionic moduli come in two varieties, since they carry a positive or negative unit of Maxwell charge. A basis is provided by $2g-2+c$ positive and $2g-2-c$ negative holomorphic $\frac{3}{2}$ -differentials, χ_j^+ and χ_k^- , respectively. It is convenient to choose these differentials delta-function localized in points z_j^+ and z_k^- on the Riemann surface.

Finally, any Maxwell field may be Hodge-decomposed into a background carrying the instanton number, an exact and a co-exact piece, plus a harmonic component $A(m_A)$ representing the Maxwell moduli. The latter lives in the complex g -dimensional space of holomorphic one-forms (or flat connections). A popular real basis consists of real one-forms $\{\alpha_i, \beta_i\}$ dual to the standard homology basis $\{a_i, b_i\}$. The Maxwell moduli space is then parametrized by the twists $\{\exp i \oint_{a_i} A(m_A), \exp i \oint_{b_i} A(m_A)\}$ around the homology cycles, showing that it is just a (real) $2g$ -torus.

3. The String Measure

Amplitudes for $N=2$ closed string scattering have now been reduced to

$$\langle \dots \rangle = \sum_{g=0}^{\infty} \kappa^{2-2g} \sum_{c=2-2g}^{2g-2} \lambda^c \int dm \langle \dots \rangle_{g,c}^{(m)} \quad (1)$$

where κ and λ are the gravitational and Maxwell string couplings, and the fixed-moduli correlator is given by

$$\langle \dots \rangle_{g,c}^{(m)} = \int D[X \psi b c \beta \gamma \tilde{b} \tilde{c}] \text{AZI} \dots e^{iS[X-\tilde{c};m]} \quad (2)$$

Here, S stands for the gauge-fixed action, and proper superconformal gauge fixing has led to the antighost zero-mode insertions

$$\begin{aligned} \text{AZI} &= \left| \prod_{\ell} \langle h_{\ell}, b \rangle \right|^2 \prod_i \langle \alpha_i, \tilde{b} \rangle \langle \beta_i, \tilde{b} \rangle \\ &\times \left| \prod_j \delta(\langle \chi_j^+, \beta^- \rangle) \prod_k \delta(\langle \chi_k^-, \beta^+ \rangle) \right|^2 \end{aligned} \quad (3)$$

using the notation $\langle \theta, \omega \rangle = \int \theta \wedge * \omega$.

The moduli $m = (m_h, m_\chi, m_A)$ appear in the action (apart from the implicit dependence in the Hodge star $*$) only linearly via

$$S(m) \sim \langle h, T \rangle + \langle \chi^\pm, G^\mp \rangle + \langle A, J \rangle \quad (4)$$

Thus, the integrals over fermionic and Maxwell moduli can be performed! The former produces a factor of

$$\left| \prod_j \langle \chi_j^+, G^- \rangle \prod_k \langle \chi_k^-, G^+ \rangle \right|^2 \quad (5)$$

which combines with part of the AZI to

$$\left| \prod_j \text{PCO}^-(z_j^+) \prod_k \text{PCO}^+(z_k^-) \right|^2 \quad (6)$$

where I defined the picture-changing operators

$$\text{PCO}^\pm(z) := \delta(\beta^\pm(z)) G^\pm(z) \quad (7)$$

for $\chi_*^\pm \sim \delta(z-z_*^\pm)$. These operators are well-known and needed. Their position dependence is BRST-trivial, so they may be moved around at will. As for the Maxwell moduli, integrating them out yields

$$\prod_i \delta(\langle \alpha_i, J \rangle) \delta(\langle \beta_i, J \rangle) \quad (8)$$

which joins another part of the AZI to form

$$\prod_i PCO^0(a_i) PCO^0(b_i), \quad (9)$$

introducing yet another kind of picture-changing operator,

$$PCO^0(\gamma) := \delta\left(\int_{\gamma} *J\right) \int_{\gamma} *\tilde{b}, \quad (10)$$

associated with a path γ . Note that it are Kronecker δ s which appear in Eqs. (8) and (10). $PCO^0(\gamma)$ depends only on the homotopy of γ . The novel insertions of PCO^0 serve to project the sum over the complete set of states flowing through a handle onto the charge-neutral sector, as factorization from pinching the handle demands.

All taken together, Eq. (1) simplifies to

$$\langle \dots \rangle = \sum_{g,c} \kappa^{2-2g} \lambda^c \int dm_h \int D[X - \tilde{c}] \mathcal{M} \dots e^{iS_0(m_h)} \quad (11)$$

where the action

$$S_0(m_h) = S[X - \tilde{c}; m_h, m_{\chi}=0, m_A=0] \quad (12)$$

still depends on g and c through the classical background metric and Maxwell field, and the string measure takes the form

$$\begin{aligned} \mathcal{M} = & \left| \prod_{\ell} \int_{\Gamma_{\ell}} b \right|^2 \prod_i PCO^0(a_i) PCO^0(b_i) \\ & \times \left| \prod_j PCO^-(z_j^+) \prod_k PCO^+(z_k^-) \right|^2 \end{aligned} \quad (13)$$

for $g \geq 2$ and $|c| \leq |2g-2|$ (sphere and torus need minor modifications).

4. Adding Punctures

The proper way to deal with string amplitudes is to consider the vertex operator insertions (\dots) as $N=2$ punctures and integrate “standard vertex operators” over the moduli space of *punctured* (super) Riemann surfaces. The term “standard” refers to the canonical representation of vertex operators, which for $N=2$ closed strings means total ghost number $u = 0$ and picture numbers $\pi^{\pm} = -1$. Since the Euler number of a n -punctured surface is $2-2g-n$, each vertex operator carries a factor of κ^{-1} . The presence of n external legs then has the following consequences.

Each puncture introduces an additional (complex) metric modulus, namely the position z_p of the puncture. The associated b -ghost zero-mode insertion $|\oint_p b|^2$ may be employed to strip the c ghosts off the standard vertex operator $V^c = c\bar{c}W^{(1,1)}(z_p)$. On the sphere or torus, conformal isometries yield three or one c zero modes, each of which cancels one b zero mode and one z_p integration, leaving three or one unintegrated c -type vertices, respectively. All other vertex locations are to be integrated over, in accordance with the complex dimension $3g-3+n$ of extended metric moduli space.

Likewise, each puncture adds one positive and one negative (complex) fermionic modulus, increasing the complex dimensions of the fermionic moduli spaces to $2g-2\pm c+n$. Repeating the computation of the previous section, this simply yields

$$\left| \prod_p PCO^-(z_p) PCO^+(z_p) \right|^2 \quad (14)$$

which may be used to convert all vertex operators to the $\pi^{\pm}=0$ picture. A notable exception is again the sphere, where two Killing spinors eliminate the fermionic moduli of two punctures.

Finally, the number of Maxwell moduli increases. The Maxwell field is allowed to develop single poles at the punctures, whose residues $\oint_p A$ and $\oint_p *A$ represent additional moduli. Their complex number is $n-1$ since the meromorphy of A forces the sum of the residues to vanish. The extra homology cycles dual to these A twists are the contours c_p around the vertex locations z_p , paired with curves connecting the punctures to a common base point z_0 . Since a contour encircling *all* vertex locations is homotopically trivial, and since one may select one vertex location as base point, only $n-1$ such pairs of curves are independent, in agreement with the counting of moduli. Like for the string measure \mathcal{M} , integrating out the Maxwell puncture moduli generates $2n-2$ insertions of PCO^0 which serve to strip the \tilde{c} ghosts off the standard vertex $V^{\tilde{c}} = \tilde{c}\bar{\tilde{c}}W^{(0,0)}(z_p)$. This leaves a single vertex operator dressed with $\tilde{c}\bar{\tilde{c}}$, exactly what is needed to neutralize the single \tilde{c} and $\bar{\tilde{c}}$ zero modes.

In total, the moduli associated with vertex punctures can be taken into account by choosing an appropriate set of picture- and ghost-number representatives for the vertex operators. Correlators fail to vanish only if their total ghost number adds up to zero and their picture numbers sum to $2g-2\mp c$.

5. Spectral Flow

A constant shift $m_A \rightarrow m_A + \delta m_A$ in the integration over the Maxwell moduli should not effect any string amplitude. On the other hand, it does contribute an additional factor of $\exp\{\frac{i}{\pi}\langle\delta A, J\rangle\}$ to the measure, with $\delta A \equiv A(\delta m_A)$. Using $d*J=0$ and the fact that δA is harmonic outside the punctures, one computes

$$\begin{aligned} \langle\delta A, J\rangle &= \sum_{i=1}^g \left(\oint_{a_i} \delta A \oint_{b_i} *J - \oint_{b_i} \delta A \oint_{a_i} *J \right) \\ &\quad + \sum_{p=1}^n \oint_{c_p} \delta A \int_{z_0}^{z_p} *J. \end{aligned} \quad (15)$$

Clearly, a residue $\text{res}_p \delta A = -\theta$ at $z_p = z$ induces a twist by the “spectral flow operator”

$$SFO(\theta, z) := \exp\left\{2\theta \int_{z_0}^z *J\right\} \quad (16)$$

at the puncture. The harmonicity of δA implies $\sum_p \text{res}_p \delta A = 0$, restricting the puncture twists to $\sum_p \theta_p = 0$. Their effect may be subsumed by twisting the vertex operators

$$V_p(z_p) \longrightarrow V_p^{(\theta_p)}(z_p) = SFO(\theta_p, z_p) \cdot V_p(z_p). \quad (17)$$

A number of interesting remarks are to be made here. First, SFO is a *local* operator, since bosonization $*J = \frac{1}{2}d\Phi$ yields [2,4]

$$SFO(\theta, z) = \lambda^{-\theta} \exp\{\theta \Phi(z)\}, \quad (18)$$

identifying the Maxwell string coupling

$$\lambda = \exp\{\Phi(z_0)\}. \quad (19)$$

Second, the dependence on the base point z_0 (and on λ) drops out in correlators exactly when the total twist vanishes as required, $\sum_p \theta_p = 0$. Third, SFO is BRST invariant but not trivially so; its *derivative*, however, is BRST trivial, so I may

move SFO around at no cost. Fourth, as a consequence, amplitudes are twist-invariant,

$$\begin{aligned} \langle V_1^{(\theta_1)} \dots V_n^{(\theta_n)} \rangle &= \langle V_1^{(0)} \dots V_n^{(0)} \Pi_p SFO(\theta_p) \rangle \\ &= \langle V_1^{(0)} \dots V_n^{(0)} SFO(\sum_p \theta_p) \rangle \\ &= \langle V_1 \dots V_n \rangle, \end{aligned} \quad (20)$$

as long as the total twist vanishes. This is nothing but Maxwell modular invariance. Fifth, $SFO(\theta = \pm \frac{1}{2})$ turns NS into R vertices and vice versa, explicitly establishing their physical equivalence. Sixth, SFO seems to ruin global $U(1,1)$ invariance since $\exp\{\theta\Phi\}$ carries Lorentz charge proportional to θ . However, from Eqs. (16-19) one concludes that λ must transform compensatingly under $U(1,1)$, since SFO is BRST-equivalent to the unit operator. Thus, the Maxwell string coupling is *not* a Lorentz scalar! Seventh, $SFO(m \in \mathbf{Z}, z)$ creates or destroys a singular instanton by way of a singular gauge transformation, $A \rightarrow A - m d \ln z$. More precisely,

$$SFO(m, z=0) |c=0\rangle = \lambda^{-m} |c=m\rangle, \quad m \in \mathbf{Z} \quad (21)$$

maps from the trivial to the m -instanton vacuum, with the c -value coming from the delta-peaked Maxwell field strength integrated over a disc centered at $z=0$. Hence, the amplitude contributions with nonzero values of c arise from imposing a total twist of $\sum_p \theta_p = c$. Eighth, the instanton-number-changing operators

$$\begin{aligned} ICO^\pm(z) &:= \lambda^{\pm 1} SFO(\theta = \pm 1, z) \\ &= \psi^{\pm,0} \psi^{\pm,1} \delta(\gamma^\pm) \delta(\beta^\pm) (1 \mp 2\tilde{b}c)(z) \end{aligned} \quad (22)$$

may be moved to the punctures, where they change the Lorentz charges and the picture assignments $(\pi_+, \pi_-) \rightarrow (\pi_+ \pm 1, \pi_- \mp 1)$ of the vertex operators. In this way, spectral flow may be used to convert any $c=m$ correlator of $U(1,1)$ singlets to a $c=0$ correlator of *Lorentz-charged* vertex operators.

To sum up, I have demonstrated how Maxwell moduli transformations correspond to the spectral flow of the $N=2$ superconformal algebra, and how the latter is implemented on correlators of vertex operators. This connection becomes even more transparent after bosonizing the spinor matter and ghost fields [2,24].

6. Amplitudes

To illustrate the arguments presented above, let me compute the leading contributions to the three- and four-point functions of the single massless scalar particle, restricting myself to one chiral half. Like for the bosonic string, the massless chiral three-point correlator must be antisymmetric in the external momenta k_a , $a=1, 2, 3$. From two on-shell momenta $k_a^{\pm, \mu}$ and $k_b^{\pm, \nu}$ one can form three antisymmetric $O(1, 1)$ invariants, namely

$$\begin{aligned} c_{ab}^0 &= \eta_{\mu\nu} k_a^{[+, \mu} k_b^{-], \nu} \\ d_{ab}^0 &= \epsilon_{\mu\nu} k_a^{\{+, \mu} k_b^{-\}, \nu} \\ c_{ab}^\pm &= \epsilon_{\mu\nu} k_a^{\pm, \mu} k_b^{\pm, \nu} \end{aligned} \quad (23)$$

of which only the first one is $U(1, 1)$ invariant.

For the simplest case of $(g, c, n) = (0, 0, 3)$ the total picture must be $(\pi_+, \pi_-) = (-2, -2)$, so a convenient choice of representatives yields

$$\begin{aligned} A_3^{(0,0)}(k_1, k_2, k_3) &= \langle V(k_1, z_1) V(k_2, z_2) V(k_3, z_3) \rangle_{(0,0)} \\ &= \langle \tilde{c} c W_{(-1,-1)}(1) c W_{(-1,0)}(2) c W_{(0,-1)}(3) \rangle_{(0,0)} \\ &= c_{23}^0 \end{aligned} \quad (24)$$

The $c=1$ contribution may be calculated from twisting, say, the third vertex operator in Eq. (24) by $\theta=1$. The same result obtains when, alternatively, I use *untwisted* vertex operators in the $(\pi_+, \pi_-) = (-3, -1)$ background:

$$\begin{aligned} A_3^{(0,1)}(k_1, k_2, k_3) &= \langle V(k_1, z_1) V(k_2, z_2) V(k_3, z_3) \rangle_{(0,1)} \\ &= \langle \tilde{c} c W_{(-1,-1)}(1) c W_{(-1,0)}(2) c W_{(-1,0)}(3) \rangle_{(0,1)} \\ &= c_{23}^+ \end{aligned} \quad (25)$$

Since $|c| \leq 1$ for the tree-level three-point function, I find a total of

$$A_3^{g=0} = \lambda^{-1} c_{23}^- + c_{23}^0 + \lambda c_{23}^+ \quad (26)$$

Taking into account the Lorentz properties of λ it appears that a singlet is formed from two $SU(2)$ triplets. This $SU(2)$ being part of the Lorentz group $U(1, 1)$, I conclude that ICO^\pm play the role of spacetime Lorentz generators!

It is well-known that the absence of any massive resonances enforces the vanishing of the four-point function [11]. Nevertheless, it is instructive to verify this claim. The $N=2$ Veneziano amplitude receives from $c=0$ the term

$$\begin{aligned} A_4^{(0,0)}(k_1, k_2, k_3, k_4) &= \langle \tilde{c} c W_{(-1,0)} \int W_{(0,-1)} c W_{(-1,0)} c W_{(0,-1)} \rangle_{(0,0)} \\ &\propto k_1^+ \cdot k_2^- k_3^+ \cdot k_4^- t + k_1^+ \cdot k_4^- k_2^- \cdot k_3^+ s \\ &\propto c_{12}^0 c_{34}^0 t + c_{14}^0 c_{32}^0 s - 16stu \\ &= 0 \end{aligned} \quad (27)$$

due to the peculiar massless kinematics in $\mathbf{C}^{1,1}$. The application of spectral flow extends this vanishing to all nonzero values of c .

The machinery to evaluate loop corrections is now in place. As an example, the one-loop three-point correlators turn out as

$$A_3^{(1,c)} \propto (c_{23}^0)^{3-|c|} (c_{23}^\epsilon)^{|c|} \quad (28)$$

with $\epsilon = \text{sign}(c) = \pm$ and $c = -3, \dots, +3$. Since all amplitudes with more than three legs should vanish for any value of g and c , only the one-, two- and three-point functions for $g \geq 2$ are yet to compute.

The expressions for $A_3^{(g,c)}$ are to be interpreted as string loop and instanton corrections to the effective spacetime action, which happens to be self-dual Yang-Mills and self-dual gravity for open and closed critical $N=2$ strings, respectively.

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